Recitation 6

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Problem 1. Put the two vectors into a matrix and row reduce it. You get

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since every column is pivotal, the two vectors are independent, and so they form a basis of $\mathbb{R}^2$.
To find coordinates of $x = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ is the basis $\mathcal{B} = \{v_1, v_2\}$ you need to find scalars c_1, c_2 such that $x = c_1 v_1 + c_2 v_2$. So you need to solve the system $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$. Then the solution is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \frac{1}{2 \cdot 1 - 1 \cdot 1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 12 \end{bmatrix}$$

Problem 2. Polynomials $\{1+t, t^2-1, t+t^2-1\}$ can be represented (relative to the basis $\{1, t, t^2\}$) by the columns $[1,1,0]^T$, $[-1,0,-1]^T$ and $[-1,1,1]^T$ respectively. So to figure out if they form a basis, we need to row reduce the matrix

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

You row reduce, see that there are 3 pivots, so the columns are linearly independent. Since $\mathbb{P}_2$ is 3-dimensional, the columns for a basis.
The polynomial $3t^2$ is represented by the column $[0,0,3]^T$. To find its coordinates in the basis $\{1+t, t^2-1, t+t^2-1\}$ we need to solve the system

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Then the solution is $[3, 6, -3]^T$.

Problem 3. The vector space V can be described as the null space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

Since there are two pivots, the null space of that matrix is $4 - 2 = 2$ dimensional.
The two vectors

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

form a basis of the space V.

Problem 4. Dimension of the column space is the rank, and so using the rank-nullity formula we get $rank(A) = 9 - 8 = 1$.

Problem 5. If A is 5×8, it defines a transformation $\mathbb{R}^8 \to \mathbb{R}^5$. If $\dim Nul(A) = 2$, nullity-rank formaula gives that $\dim Col(A) = 8 - 2 = 6$. However, $Col(A)$ is a subspace of $\mathbb{R}^5$, so it can't be more than 5-dimensional.

Problem 6. The spaces $Col(A)$ and $Row(A)$ are different in general. Take, for example, matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

However, the dimensions of these spaces are always the same.

Problem 7. In $\mathbb{R}^n$: $Row(A), Nul(A), Col(A^T)$.
In $\mathbb{R}^m$: $Col(A), Row(A^T), Nul(A^T)$.
Problem 8. Row reducing gives

$$\begin{bmatrix} 1 & a \\ a & a+6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & a \\ 0 & -a^2+a+6 \end{bmatrix}$$

The second column will be pivotal if and only if $-a^2 + a + 6 \neq 0$, i.e. $a \neq 3$ and $a \neq -2$.

Problem 9. Variables x_1, x_3 are basic, variable x_2 is free. There are two basic variables (two pivots) so $rank(A) = 2$. Solving the system, we have $x_3 = 2, x_1 = -3x_2 + 2x_3 - 1 = 3 - 3x_2$, and x_2 is free.
Since the second column is a multiple of the first one in the reduced matrix, this was also the case in the original matrix. Therefore, the second column of the original matrix is

$$\begin{bmatrix} -9 \\ 6 \\ 3 \end{bmatrix}$$

Problem 10. Row reduce the matrix, see that there are 3 pivots, and so the matrix is invertible.
If you have are solving the system $Ax = b$ and you know A^{-1}, then $x = A^{-1}b$ is the solution (notice the order of multiplication).
To find the inverse of A, need to put A into the augmented matrix $[A|I]$ and row reduce to the max. The inverse turns out to be

$$A^{-1} = \begin{bmatrix} 5/2 & -5 & -17/4 \\ 1/2 & -2 & -7/4 \\ -1/2 & 1 & 3/4 \end{bmatrix}$$

Problem 11. You need to figure out if there are two scalars c_1, c_2 such that

$$\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 5 \\ -5 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

So you need to see if the system

$$\begin{bmatrix} 5 & 1 & | -1 \\ -5 & 1 & | \;\; 3 \\ 4 & 2 & | \;\; 2 \end{bmatrix}$$

is consistent. Row reduce, every column turns out to be pivotal. Since there is a pivot in the last column of the augmented matrix, the system is inconsistent, and so b is not in the $Col(A)$.